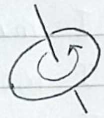
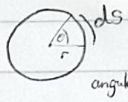
Unit 3: Rotation and Gravity

Lecture 1

Rotating Disk

All points rotate the same amount.





$\frac{ds}{r} = \theta$

angular vel: $\omega = \frac{\Delta\theta}{\Delta t}$ angular acc: $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Suvat for rotation

$\omega = \omega_0 + \alpha t$

$\Delta\theta = \omega_{av}\Delta t$

$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega_{av} = \frac{1}{2}(\omega_0 + \omega)$

$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

Linear velocity $V_{tang} = \frac{ds}{dt}$ and $s = r\,d\theta$

$V_{tang} = r\frac{d\theta}{dt} = r\omega$

hence

tangential velocity $\rightarrow a_t = r\frac{d\omega}{dt} = r\alpha$ not $r\omega^2$.

+ acceleration will change depending on r.

Centripetal acceleration: $a_c = r\omega^2 = \frac{V_t^2}{r}$ <- - - - centripetal

The total acceleration $a_t = \sqrt{a_c^2 + a_t^2}$ pythag since $a_c = 90°$ to a_t

Torque

The only way to spin a disk is to add a tangential force.

Torque is the rotational component of a force.

$\tau = rF_t$ ← tangential component of force

$\tau = rF\sin\phi$ to resolve   it's like a moment.

Moment of Inertia

Its like torque (moments) but focused on mass.

$I = \sum m_i r_i^2$ Sum of all mr^2 in system

$I = \int r^2\,dm$

Bringing it all together!

$\tau = rF_t = rma_t$ because $F = ma$

$\tau = mr^2\alpha$ because $a_t = r\alpha$

So for 1 system of many objects:

$\tau_{net} = \sum\tau = \sum mr^2\alpha$ but $\sum mr^2 = I$

$\tau_{net} = I\alpha$

Newton's 2nd Law of Rotation

Inertia formulae for more than a disk:

hollow cylinder (disk) $I = mr^2$

Solid cylinder $I = mr^2/2$

Solid sphere $I = 2mr^2/5$

$$I = \int r^2 dm \rightarrow I = \int r^2 \rho dV$$

Parallel axis Theorem (Steiner theorem)

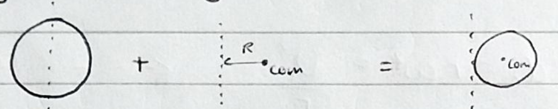
All the above applies when rotation is through the centre of mass.

if not: $I = I_{com} + Mh^2$

rotation around COM

rotation at a distance from COM to rotation axis (h = distance from COM to axis)

Eg. sphere rotating from edge



I_{com}	Mh^2	
$\frac{2}{5}MR^2$	MR^2	$\frac{7}{5}MR^2$.

Rotational Energy:

kinetic energy: $KE = \frac{1}{2}mv_t^2 = \frac{1}{2}mr^2\omega^2$ $\quad v_t = r\omega$

$= \frac{1}{2}\omega^2 \sum mr^2$ ← look! Inertia.

$= \frac{1}{2}I\omega^2$

Lecture 2

$Work = \tau\, d\theta$

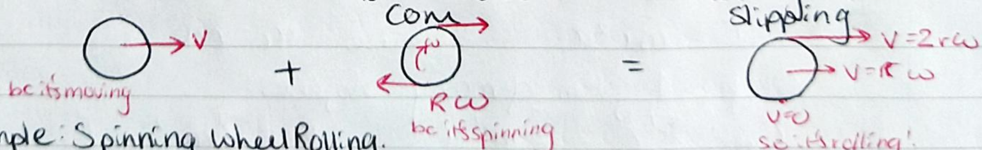
$Power = \tau\omega$

As a ball / disc rotates, the surface moves at $V_t = r\omega$

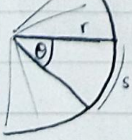
Rotation without slipping - the bottom point of ball (bit in contact with the ground) must be momentarily at rest.

If it slips too, ~~add on a translational velocity~~ we'll cover later.

Translation v | Rotation about COM | Rolling w/o Slipping



Example: Spinning Wheel Rolling.



$s = r\theta$

$v = \frac{ds}{dt} = r\omega$ in non-slip conditions at centre of mass

$a = r\alpha$

$V_{top} = 2V$

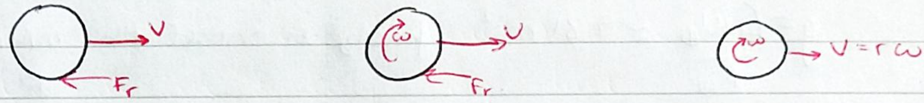
$V_{bottom} = 0$

The total kinetic energy of a rolling object

Translation		Rotation		total
$\frac{1}{2}mv^2$	+	$\frac{1}{2}I\omega^2$	=	KE

Rolling with Slipping

Example: Bowling ball



when you throw it, it starts w/ just gliding on floor

Friction ↑ω but ↓v until bottom is 0

Now it's rolling w/o slipping! when $v = r\omega$, only rotation (rolling w/o)

When do we make the transition?

we know: $v = v_0 + at$

$f_r = \mu Mg = -Ma$ ← deceleration

R

$a = -\mu g$

$v = v_0 - \mu gt$

$\omega = \omega_0 + \alpha t$

$\tau = I\alpha = \frac{2}{5}Mr^2\alpha$ solid sphere

$\tau = f_r R$ ← it's a moment

$\alpha = \frac{5\mu g}{2r}$ some rearranging

Our non-slip condition $v = r\omega$

$\omega = 5\mu gt/2r$ sub into $\omega = \omega_0 + \alpha t$

$v = r\frac{5\mu gt}{2r} = \frac{5}{2}\mu gt = v_0 - \mu gt$

← rearrange these 2

so $t = \frac{2v_0}{7\mu g}$

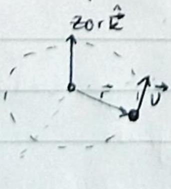
and $v_{com} = \frac{5v_0}{7}$ & $s = \frac{12v_0^2}{49\mu g}$ solid sphere only!!!

Angular Momentum

$\vec{L} = \vec{r} \times \vec{p}$ and $\vec{p} = m\vec{v}$

$\vec{L} = \hat{r}mv\sin\theta\hat{k}$ ← unit vector for z axis.

Example special case: circular orbit



r & v are perpendicular ∴ θ = 90°

$\vec{L} = rmv\sin 90°\hat{k} = rmv\hat{k}$

$v = r\omega$ from before

$\vec{L} = r^2\omega m\hat{k}$

somehow ω is in the same direction as k̂

$\vec{L} = I\vec{\omega}$

so $\omega\hat{k} = \vec{\omega}$

Angular momentum is also conserved

$\frac{d\vec{L}}{dt} = \vec{\tau}_{external}$ so if $\vec{\tau}_{external} = 0$ $\vec{L}$ = constant.

Lecture 3

Gravity

Astronomers noticed: $a_c r^2 \approx 1.33$ we'll call C_s

$$F_{sp} = m_p a = m_p \frac{C_s}{r^2} \qquad \& \quad F_{sp} = -F_{ps} \leftarrow \text{3rd law}$$

$$m_p \frac{C_s}{r^2} = m_s \frac{C_p}{r^2}$$

$$F = -\frac{G m_1 m_2}{r^2}$$

$$g = \frac{G M_\oplus}{R_\oplus} \approx 9.81\ ms^{-2}$$ for objects on the Earth $mM_\oplus \approx M_\oplus$, $R \approx R_\oplus$

Escape Velocity: $\frac{GMm}{r^2} = \frac{mv^2}{r}$ $\quad U_0 + K_0 = U_f + K_f \Rightarrow 0 = U_0 + K_0$
$= \frac{1}{2}mv^2 - \frac{GMm}{r}$

$$V = \sqrt{\frac{GM}{r}} \qquad V_{escape} = \sqrt{\frac{2GM}{r}}$$

Potential Energy $U(r \to \infty) = 0$ as $F(r \to \infty) = 0$

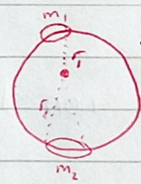
$$U = \int_\infty^r F \cdot dr = -\frac{GMm}{r}$$

Gravitational Field $g = F/m$ so $g(r) = \frac{GM}{r^2}$

Fields for a uniform shell (Hollow Earth)

On the outside: $\vec{g} = -\frac{GM}{r^2}\hat{r}$

But inside: $\vec{g} = 0$ all the fields cancel out



small r = small m

all balances

$$\frac{m_1}{m_2} = \frac{r_1^2}{r_2^2} \qquad \frac{m_1}{r_1^2} = \frac{m_2}{r_2^2}$$ and $g_1 = g_2$

forces cancel & $g = 0$.

For a sphere

outside: $g_r = -\frac{GM}{r^2}$ as usual

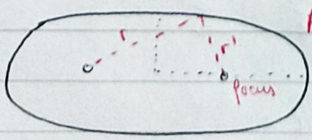
inside: $g_r = -\frac{GM'}{r^2}$ Only mass inside counts

$$M' = \rho \frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3}$$ — full radius

Lecture 4

Kepler's Laws

Law 1: All objects (in space) orbit in an ellipse, hyperbola, or parabola.
↑ closed objects



An ellipse must have $r + r' = 2a$ $\quad a = $ sma

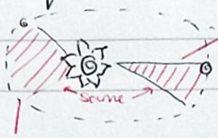
hence $a = \frac{r + r'}{2}$

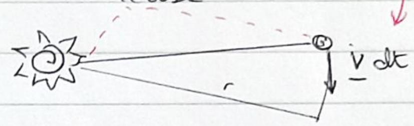
eccentricity $\varepsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2}$ and $0 < \varepsilon < 1$

on a graph $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Kepler's 2nd Law

A line joining both celestial bodies sweeps equal areas in equal times.





if dt is small, then the area ≈ ½ a parallelogram made by $\underline{r}$ a $\underline{v}\,dt$

$$dA = \tfrac{1}{2}|\vec{r} \times \vec{v}\,dt|$$

$$\frac{dA}{dt} = \tfrac{1}{2}|\vec{r} \times \vec{v}|$$ ← familiar? acceleration?

$$= \tfrac{1}{2}\frac{L}{2m}$$

Since L is conserved, dA/dt is also,

∴ Same Area.

Kepler's 3rd Law

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$$

For the solar system, T_1 a r_1 can be for Sol hence it will be a constant i.e.

$$T^2 = C_0 r^3$$

For Circular Orbits like Neptune:

$$F_g = F_c$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \frac{GM}{r} = v^2 \quad (1)$$

$$T = \frac{2\pi r}{v} \qquad \left(\frac{T^2}{4\pi^2 r^2}\right)^{-1} = v^2 \quad (2)$$

$$(1) = (2) \quad \left(\frac{T^2}{(2\pi r)^2}\right)^{-1} = \frac{GM}{r} \qquad T^2 = \frac{4\pi^2 r^2}{GM} r = \frac{(2\pi)^2}{GM} r^3$$

$$T^2 \propto r^3$$