Unit 3: Boundaries

Lecture 1

~ over a symbol = complex wavefunction.
Recap from unit 2:

$$k^2 = \underline{k} \cdot \underline{k} = \omega^2 \varepsilon_r \mu_r \varepsilon_0 \mu_0 = n^2 \frac{\omega^2}{c^2} = n^2 k^2$$

$$\boxed{v \equiv \frac{\omega}{|\underline{k}|} = \frac{1}{\sqrt{\varepsilon_r \mu_r \varepsilon_0 \mu_0}} = \frac{c}{\sqrt{\varepsilon_r \mu_r}}}$$

$\sqrt{\varepsilon_r \mu_r} = n$.

in a vacuum:
$n = 1$
$k_0^2 = \omega^2 / c^2$
$\lambda = 2\pi / k_0$

in a dielectric
$n = \sqrt{\varepsilon_r \mu_r}$ ($\mu_r = 1$ and $\sigma = 0$)
$\varepsilon_a = \varepsilon_r \varepsilon_0$ & for μ.
speed is c/n, not just c
$\lambda = 2\pi / k = 2\pi / nk_0 = \lambda_0 / n$

Still in a dielectric

$$|\underline{\tilde{B}}_0| = \frac{|\underline{\tilde{E}}_0|}{v}$$

this comes from Faraday's law:

$$\underline{\tilde{B}}_0 = \frac{\underline{k} \wedge \underline{\tilde{E}}_0}{\omega} = \frac{\hat{\underline{k}} \wedge \underline{\tilde{E}}_0}{v}$$

hence its velocity being the variable that affects amplitudes.

$$|\underline{\tilde{H}}_0| = \frac{|\underline{\tilde{B}}_0|}{\mu_r \mu_0} = \frac{|\underline{\tilde{E}}_0|}{\mu_r \mu_0 v} = |\underline{\tilde{E}}_0| \sqrt{\frac{\varepsilon_r \varepsilon_0}{\mu_r \mu_0}} = \frac{|\underline{\tilde{E}}_0|}{|Z|}$$ impedance

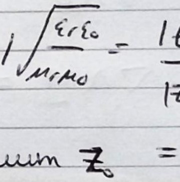
Hence
$$\boxed{|Z| = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} \simeq Z_0 / n}$$

in a vacuum $Z_0 = 377\,\Omega$

Example: microwave transmitter. Carrier frequency 6 GHz protected by dome w/ $\varepsilon_r = 3$ find n, λ_0, λ, Z.

1. $n = \sqrt{\varepsilon_r \mu_r} = \sqrt{3}$
2. $\lambda_0 = c/f = 0.05\,\text{m}$
3. $\lambda = \lambda_0 / n = 0.029\,\text{m}$ (3dp)
4. $Z = Z_0 / n = 218\,\Omega$

Boundary Conditions for a $\underline{B}$ field



let's ponder $\text{div}\,\underline{B} = 0$:
using Gauss' theorem:

$$\oint_{box} \nabla \cdot \underline{B}\, dv = \oint_{box} \underline{B} \cdot d\underline{S} = \int \underline{B} \cdot d\underline{S} \text{ of all sides.}$$

$$0 = B_{1\perp} S - B_{2\perp} S \Rightarrow B_{1\perp} = B_{2\perp}$$

Boundary Conditions for $\underline{E}$ fields

$\int \underline{E} \cdot d\underline{l} = \int_{S} \frac{\partial B}{\partial t} dS$ as thickness $\to 0$ BC and DA $\to 0$

AB
BC
CD
DA

$(E_{1\parallel} - E_{2\parallel}) \Delta l = 0$

parallel

Lecture 2

Specular Surface / Fresnel Surface

Strong reflection. Roughness < 1λ of light. water is an example, but wind can increase roughness, so no mirror image

Diffusive / Lambertian Surface

very rough. Angular distribution of light independent of illumination angle. it has same brightness all around / equally bright in all directions. Mar's surface is a good example!

Most things will be halfway-ish. Or Specular in 1 γ band, but diffuse in another because of the whole $< 1\lambda$ thing.

Tangential Components of $\underline{E}$ & $\underline{B}$ field strengths are Continuous at boundaries.

$$\nabla \wedge \underline{E} = -\frac{\partial B}{\partial t} \qquad \underline{E}_{\parallel} \text{ continuous}$$

$$\nabla \wedge \underline{H} = j_f + \frac{\partial D}{\partial t} \qquad H_{\parallel} \text{ continuous}$$

Dielectric Properties

Homogeneous: translational symmetry
Isotropic: rotational symmetry
Linear: μ & $\mathcal{E}$ constant
Sourcefree: ρ is 0
Non-conducting: $\sigma = 0$, hence $\underline{J} = 0$

If $\underline{\mathcal{E}}$ is continuous (or $\underline{B}$), $\underline{D}$ can't be continuous (or $\underline{H}$).

why?

1 medium: $\boxed{D_1 = \mathcal{E}_{r1} \mathcal{E}_0 \underline{E}_1}$

2 medium: $D_2 = \mathcal{E}_{r2} \mathcal{E}_0 \underline{E}_2$

$$\underset{\text{Continuous}}{\underline{E}_{1\parallel} = \underline{E}_{2\parallel}} \Rightarrow \frac{D_{1\parallel}}{\mathcal{E}_{r1}} = \frac{D_{2\parallel}}{\mathcal{E}_{r2}} \Rightarrow D_{1\parallel} \neq D_{2\parallel}$$

not the same

$\mathcal{E}_r$ is discontinuous, so maxwells eq produce a singularity for $\nabla \wedge \underline{D}$.

$\underline{D}$ is electric displacement of $\underline{E}$
$\underline{H}$ is intensity of $\underline{B}$

So we know
in a HHL General

$D_{2\perp} = D_{1\perp}$ & $B_{1\perp} = B_{2\perp}$ but $E_{2\parallel} = E_{1\parallel}$ & $H_{1\parallel} = H_{2\parallel}$

in a HIL

$D_{2\perp} = D_{1\perp}$ $\qquad \frac{1}{\epsilon_2} D_{2\parallel} = \frac{1}{\epsilon_2} D_{1\parallel}$

$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$ $\qquad E_{2\parallel} = E_{1\parallel}$

$B_{1\perp} = B_{2\perp}$ $\qquad \frac{1}{\mu_1} H_{1\parallel} = \frac{1}{\mu_2} H_{2\parallel}$

$\mu_1 H_{1\perp} = \mu_2 H_{2\perp}$ $\qquad H_{2\parallel} = H_{1\parallel}$

Plane wave normal reflection for Dielectrics

incident

$\underline{\tilde{E}}_I(z,t) = \tilde{E}_{01} e^{i(k_1 z - \omega t)} \hat{\underline{x}}$

$\hat{B}_1(z,t) = \tilde{B}_{01} e^{i(k_1 z - \omega t)} \hat{y}$

Reflected

$\hat{E}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\underline{x}}$

$\tilde{B}_R(z,t) = \tilde{B}_{0R} e^{i(-k_1 z - \omega t)} \hat{x}$

$\tilde{\underline{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{\underline{x}}$

$\hat{\underline{B}}_T(z,t) = \tilde{B}_{0T} e^{i(k_2 z - \omega t)} \hat{y}$

Transmitted

Hence:

$$\boxed{\begin{aligned} (\tilde{\underline{E}}_1 + \hat{\underline{E}}_R)_{\parallel} &= (\hat{\underline{E}}_T)_{\parallel} \\ (\hat{\underline{H}}_1 + \hat{\underline{H}}_R)_{\parallel} &= (\hat{\underline{H}}_T)_{\parallel} \end{aligned}}$$

at boundary $z=0$ $(\tilde{\underline{E}}_{01} - \tilde{\underline{E}}_{0R} = \tilde{\underline{E}}_{0T})$

$(\hat{\underline{H}}_{01} - \hat{\underline{H}}_{0R} = \hat{\underline{H}}_{0T})$

$n_1(\hat{E}_{01} - \hat{E}_{0R}) = n_2 \hat{E}_{0T}$ $\qquad \frac{\hat{E}_{01}}{Z_1} - \frac{\hat{E}_{0R}}{Z_1} = \frac{\hat{E}_{0T}}{Z_2}$ impedance.

At Normal Incidence

using equations as above:

reflection coefficient: $\qquad r \equiv \frac{\tilde{E}_{0R}}{\tilde{E}_{01}} = \frac{n_1 - n_2}{n_1 + n_2}$

transmission coefficient $\qquad t \equiv \frac{\hat{E}_{0T}}{\tilde{E}_{01}} = \frac{2n_1}{n_1 + n_2}$

Fresnel Coefficients.

Reflectance

Not Reflectivity!! Reflectance = $\dfrac{\text{flux reflected}}{\text{flux impinging}} = \boxed{\dfrac{F_R}{F_I} = R}$

$$R \equiv \frac{\langle \underline{S}_R \rangle \cdot \hat{n}}{\langle \underline{S}_I \rangle \cdot \hat{n}} = \frac{0.5\, E_{0R} H_{0R}}{0.5\, E_{0I} H_{0I}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \text{ hence } \boxed{R = r^2}$$

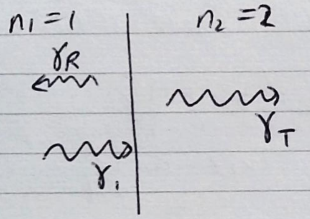
Transmittance

Transmittance = $\dfrac{\text{transmitted flux}}{\text{incident flux}} = \boxed{\dfrac{F_T}{F_I}}$

$$T \equiv \frac{\langle \underline{S}_T \rangle \cdot \hat{n}}{\langle \underline{S}_I \rangle \cdot \hat{n}} = \frac{0.5\, E_{0T} H_{0T}}{0.5\, E_{0I} H_{0I}} = \frac{Z_1}{Z_2}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{n_2}{n_1}\left(\frac{n_1}{n_1 + n_2}\right)^2$$

$$\boxed{T = \frac{n_2 t^2}{n_1}}$$

example



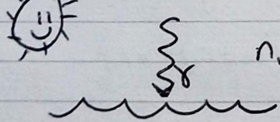
$R = \left(\dfrac{1-2}{1+2}\right)^2 = 1/9$

$T = \left(\dfrac{8}{1+2}\right)^2 = 8/9$

and $\boxed{T + R = 1}$ wow!

Example

Earth's ocean



$n_2 = 1.33$

so $R = \left(\dfrac{0.33}{2.33}\right)^2 = 0.02$

$T = 0.98$ so hardly any light is reflected.

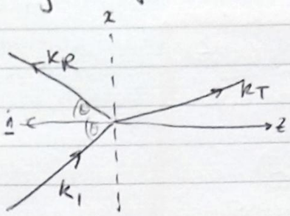
This is why ocean looks dark blue.

Interestingly Ice's Structure makes its $n \approx n_{air}$ so it looks whiter and more reflective.

Lecture 3

lets look at a ray diagram

$\boxed{K_R = n_1 \frac{\omega_R}{c}}$



$\boxed{K_T = n_2 \frac{\omega_T}{c}}$

due to symmetry $K_{Ry} = k_{Ty} = 0$.

$\boxed{k_1 = n_1 \frac{\omega_1}{c}}$

So parallel $\underline{E}$ is continuous.

$\underline{k} \cdot \underline{r} = k_{1x} x + k_{1z} z$ $\quad n_1 \quad\quad n_2$

hence:

$$[\tilde{\underline{E}}_{01}]_{x,y} e^{i(k_{1x}x - \omega_1 t)} + [\hat{\underline{E}}_{0R}]_{x,y} e^{i(k_{Rx}x - \omega_R t)} = E_{Transmitted}$$

same writing

for this to be true:

$$\boxed{\omega_1 = \omega_R = \omega_T} \qquad K_{1x} = K_{Rx}$$

and $\underline{k} \cdot \underline{k} = k_1^2 = K_R^2 = \frac{\omega^2}{V_1^2}$

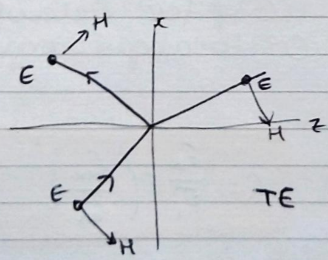
implying $\Theta_1 = \Theta_R$ & also $k_{1x} = k_{Tx}$

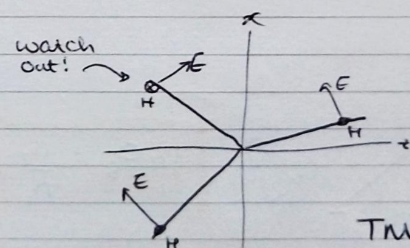
hence

$$\frac{\sin\Theta_T}{\sin\Theta_1} = \frac{k_1}{k_T} = \frac{n_1}{n_2} \qquad \boxed{n_1 \sin\Theta_1 = n_2 \sin\Theta_2}$$

Now lets add Es and Bs

TE - E field comes out of paper Transverse Electric
TM - $\underline{B}$ field comes out of paper Transverse Magnetic





for TM:

in x component:

$(E_{01} + E_{0R})\cos\Theta_1 = E_{0T}\cos\Theta_T$

$H_{01} - H_{0R} = H_{0T}$ $\quad$ but $Z = \frac{E}{H}$ so $\frac{E_{01}}{Z_1} - \frac{E_{0R}}{Z_1} = \frac{E_{0T}}{Z_2}$

one points inwards

reflection coefficient $r_{TM} = \left(\frac{E_{0R}}{E_{01}}\right)_{TM} = \frac{Z_2\cos\theta_T - Z_1\cos\theta_1}{Z_2\cos\theta_T - Z_1\cos\theta_1}$

$$\boxed{\begin{aligned} r_{TM} &= \frac{n_1\cos\theta_T - n_2\cos\theta_1}{n_1\cos\theta_T + n_2\cos\theta_1} \\ t_{TM} &= \frac{2n_1\cos\theta_1}{n_1\cos\theta_T + n_2\cos\theta_1} \end{aligned}}$$

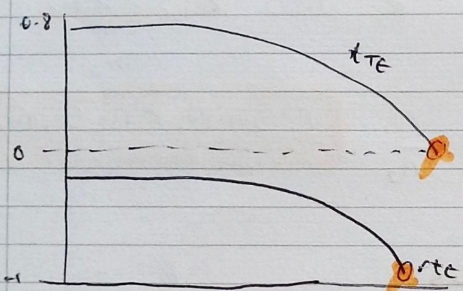
For TE waves

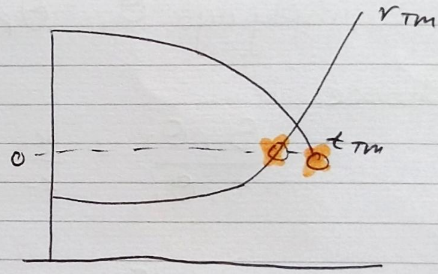
y comp $\hat{}$ $E_{01} + E_{0R} = E_{0T}$

$H_{01}\cos\theta_1 - H_{0R}\cos\theta_R = H_{0T}\cos\theta_T$

$\frac{E_{01}}{Z_1}\cos\theta_1 - \frac{E_{0R}}{Z_1}\cos\theta_1 = \frac{E_{0T}}{Z_2}\cos\theta_T$

$$\boxed{\begin{aligned} r_{TE} &= \left(\frac{E_{0R}}{E_{01}}\right)_{TE} = \frac{n_1\cos\theta_1 - n_2\cos\theta_T}{n_1\cos\theta_1 + n_2\cos\theta_T} \\ t_{TE} &= \left(\frac{E_{0T}}{E_{01}}\right)_{TE} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_T} \end{aligned}}$$

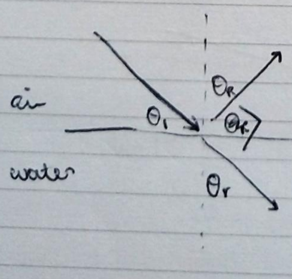




so 180° φ all angles

Brewster angle
= where reflected field is 0.
180° φ for below Brewsters'
0° for larger angles

Lecture 4



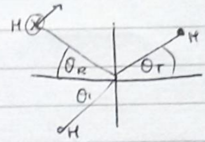
$$\boxed{\begin{aligned} \tan\theta_B &= \frac{n_2}{n_1} \\ \theta_I &= 90^\circ - \theta_B \\ &= \pi/2 - \theta_B \end{aligned}}$$

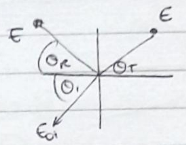
$$\boxed{R = \frac{F_R}{F_1} = \left(\frac{E_{0R}}{E_{01}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

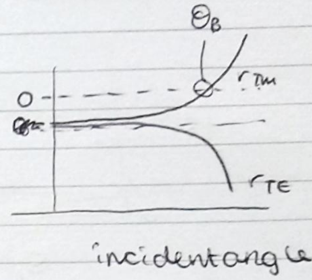
Fresnel Coefficients for E field

$$r_{TM} = \frac{n_1 \cos\theta_T - n_2 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}$$

$$r_{TE} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_T}{n_1 \cos\theta_1 + n_2 \cos\theta_T}$$







For TM Waves Specifically:

$r_{TM} = 0$ at $\theta_1 = \theta_B$ ← brewster angle

<u>only at θ_B</u>

so $\theta_1 = \pi/2 - \theta_B$ & $\theta_B = \text{arctangent}\left(\frac{n_2}{n_1}\right)$

This **never** happens for TE wave.

Example: air-glass

air → glass

$$\theta_B = \text{arctangent}\left(\frac{n_2}{n_1}\right) = \arctan 1.5 = 56.31^\circ$$

$$\theta_1 = 90^\circ - 56.31 = 33.7^\circ$$

glass → air

$$\theta_B = \text{arctangent}\left(1/1.5\right) = 33.7^\circ$$

} wow!

This is because $\text{arctangent}\ 1/x = 90 - \arctan x$.

For TE Waves. reflection coefficient never $\cancel{\neq} = 0$.

$n_1 < n_2$ or $\theta_1 > \theta_T$ $\quad r_{TE} < 0$

$n_1 > n_2$ or $\theta_1 < \theta_T$ $\quad r_{TE} > 0$.

Reflectance & Transmittance

$$\boxed{R \equiv \frac{F_R}{F_I} = r^2}$$

so just sub in $(r_{TM})^2$ or $(r_{TE})^2$.

$$T \equiv \frac{F_T}{F_I} \neq t^2$$ but $\boxed{T = 1 - r^2}$

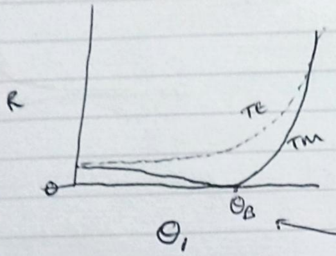
look at lecture slides for more details w/ formulae.

since $R + T = 1$

R & T from 0 to 1 each

but only if $|r| \cancel{\geq} 1$

Reflectance in more detail:



$R_{normal} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{0.5}{2.5}\right)^2 = 0.04$ for air $\rightarrow$ glass

reflected field is 0, because $R=0$.

θ_B = arctangent $1.5 = 56.3°$

$R_{TE} > 0$ for all θ

$R_{TE} \geq R_{TM}$ for all θ

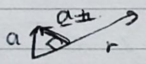
Physical Explanations.

TM: Reflected $\underline{E}$ and oscillating dipole excited by γ_T are ~~not~~ not parallel but form an angle resulting in lower reflection.

at θ_B this angle $= 90°$, hence no reflection!

TE: Reflected $\underline{E}$ and oscillating dipole vector parallel $\rightarrow$ high reflection.

Radiation Emitted by an Oscillating Dipole.



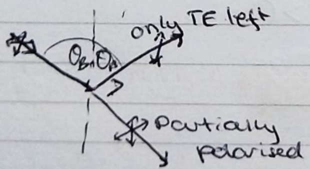
$$\boxed{E_{radiated}(r,t) = \frac{q}{4\pi\varepsilon_0} \frac{a_\perp (t - r/c)}{rc^2}}$$

$\rightarrow$ perpendicular ~~acceleration~~ vector. not acceleration.

EM is generated from <u>accelerating</u> electrical charges

The radiated E field is 0 along the oscillation direction, maximum along normal to dipole oscillation.

Polarisation Via Reflection



for water $\theta_B = \tan^{-1} 1.33 = 53°$

at Brewster angle the reflected wave is TE.

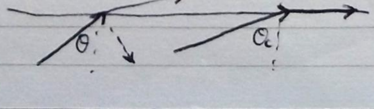
lecture 5

TIR & Evanescent Waves

TIR - angle great enough to reflect wave.

$$\boxed{\sin\theta_c = \frac{n_2}{n_1}}$$

from high n_1 to lower n_2



For TIR, this eq breaks down because $\sin > 1$.

Start from Euler's: $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$

$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$

use addition formulae:

$z = a + ib$ $\qquad\qquad \cosh^2 - \sinh^2 = 1$

$\sin(a+ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)$

$\cos(a+ib) = \cos(a)\cosh(b) - i\sin(a)\sinh(b)$

· Example: Solve in complex domain, $\sin(z) = 2$ no real sol^n

$$2 = \sin(a)\cosh(b) + \underbrace{i\cos(a)\sinh(b)}_{=0} = 2$$

$a = \pi/2$ so $\cosh b = 2$

$z = \pi/2 \pm i \,\mathrm{arccosh}(2) = \frac{\pi}{2} \pm i\,1.317$

Looking back at fresnel: $\qquad$ in TIR $\theta_1 > \theta_c$

with $r = n_1 \cos_1 - n_2 \cos\theta_T \ldots$

In TIR $\cos\theta_T$ isn't defined in real domain

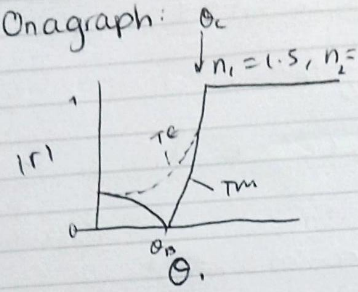
So instead $\boxed{\theta_T = \pi/2 \pm i\,\mathrm{arccosh}\left[\frac{n_1}{n_2}\sin\theta_1\right]}$.

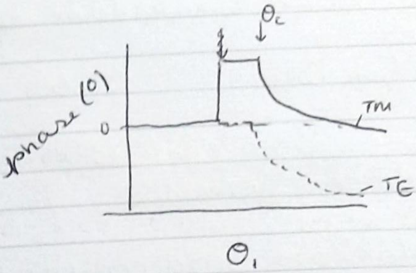
$\boxed{\sin\theta_T = \frac{n_1}{n_2}\sin\theta_1}$ real part $\qquad$ imaginary part

$$\cos\theta_T = \sqrt{1 - \sin^2\theta_T} = \boxed{i\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1 - 1}}$$

So now
$$r_{TE} = \frac{n_1\cos\theta_1 - in_2\sqrt{\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_1 - 1}}{\underbrace{n_1\cos\theta_1}_{u} + i\underbrace{n_2\sqrt{\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_1 - 1}}_{v}}$$

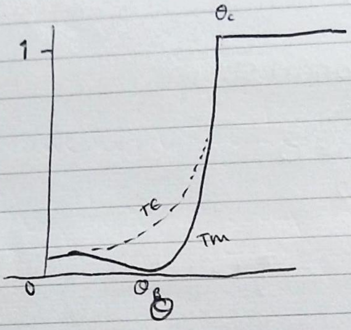
Hence $r_{TE} = \frac{u - iv}{u + iv}$ $\qquad$ just swap us and vs for TM.

On a graph: 



So θ_B affects TM only
θ_c affects both.

And Reflectance $R = |r^2|$



So all power is reflected after θ_c.

Lecture 6

In the example of glass-air

$\sin\theta_c = \frac{n_2}{n_1}$ $\quad \theta_c = 41.8°$

$\tan\theta_B = \frac{n_2}{n_1}$ $\quad \theta_B = 33.7°$

Optical Fibers

$n_{core} > n_{cladding}$ for TIR

- Core = thin glass
- cladding = surrounds core
- Buffer coating = plastic coating

Signal Degradation occurs from imperfect glass

Best ones have < 10% / km light loss.

Inhomogeneous Waves

(propagating in 1, attenuating in another).

$\underline{E}_T = \underline{E}_{0T} e^{i(\underline{k}\cdot\underline{r} - \omega t)}$ & $\hat{k}_T = k_0 (n_1 \sin\theta_1 \hat{\underline{x}} + i\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\ \hat{\underline{z}})$

$\hat{\underline{k}}_T = \tilde{\underline{k}}_{T_R} + i\tilde{\underline{k}}_{T_i}$

real $\quad$ imaginary

$k_{T_r} = n_1 \sin\theta_1\ k_0 (1, 0, 0)$

$k_{T_i} = \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\ k_0 (0, 0, 1)$

$$\tilde{E}_T = \tilde{E}_{0T} e^{i(\tilde{k}_T \cdot r - \omega t)} = E_{0T} \underbrace{e^{-k_0 \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\, \hat{z}}}_{e^{-z/l} \text{ decay}} e^{i(n_1 k_0 \sin\theta_1 x - \omega t)}$$

wave propagates in x.
wave decays in z (exponentially)

Carries no energy from surface (so $R=1$, $T=0$)
Called evanescent wave.

How fast is it decaying?

$$\boxed{l_{decay} = \frac{1}{k_0\sqrt{n_1^2 \sin^2\theta - n_2^2}} = \frac{1}{2\pi\sqrt{n_1^2 \sin^2\theta - n_2^2}}}$$

Example

find sine & cosine of transmission angle. $n_1 = 1.5$ $\theta_c = 41.8°$
to air with $\theta_1 = 60°$

$\sin\theta_T = 1.5 \sin\theta_1 = 1.3$

$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = +i\sqrt{\sin^2\theta_T - 1} = +i\, 0.829$

Find r

$r = n_1 \cos\theta_1 - \ldots = -0.1 - 0.9950i = e^{-i95.7°}$ TE

$n_1 \cos\theta_T - \ldots = 0.7217 + 0.6922i = e^{43.8°}$ TM

$t = ?$

$t = \frac{2n_1 \cos\theta_1}{n_1 \ldots} = 0.9 - 0.995i = 1.34e^{-i47.9°}$ TE

$\frac{2n_1 \cos\theta_1}{n_1 \cos\theta_T \ldots} = 0.4174 - 1.0382i = 1.12e^{-68.1}$ TM

decay distance = ?

$$\boxed{l_{decay\ evanescent} = \frac{\lambda_0}{2\pi\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}}} = \frac{\lambda_0}{5.21}$$

$$\underline{\tilde{E}}_{0T} = \tilde{E}_{0T}\,\hat{y} = \tilde{E}_{0I}\,\hat{y}\,t_{TE}\theta_I$$

$$\boxed{\underline{\tilde{k}}_T \wedge \underline{\tilde{E}}_{0T} = \omega \underline{\tilde{B}}_{0T}}$$

$$\tilde{B}_{0T} = \frac{n_2 k_0}{\omega}(\sin\theta_T\,\hat{\underline{x}} + \cos\theta_T\,\hat{\underline{z}}) \wedge \tilde{E}_{0T}\,\hat{y}$$

$$\boxed{= \frac{n_2 k_0}{c}(\sin\theta_T\,\hat{\underline{z}} - \cos\theta_T\,\hat{\underline{x}})}$$

$$\boxed{\tilde{H}_{0T} = \frac{n_2 \tilde{E}_{0T}}{Z_0}(\underbrace{\sin\theta_T\,\hat{\underline{z}}}_{\text{real}} - \underbrace{\cos\theta_T\,\hat{\underline{x}}}_{\text{imaginary}})}$$

$$\underline{\tilde{k}}_T \cdot \tilde{H}_{0T} = 0$$

Evanescent waves are inhomogeneous.

They accompany TIR.

Lecture 7

Electromagnetism Core Exercises.

1) $$E_x(x,y,z,t) = E_{0x} e^{i(k_x x + k_y y + k_z z - \omega t)}$$

a) $\frac{\partial E}{\partial t} = ?$ $\quad \frac{\partial \underline{E}}{\partial t} = -i\omega \underline{E} \quad = -i\omega E_x$

b) $\nabla \cdot \underline{E} = ?$ $\quad \frac{\partial E_x}{\partial x} = ik_x E_x \quad = i\underline{k} \cdot \underline{E}$.

c) $\nabla \wedge \underline{E} = ?$ just x. $\quad (ik_y E_z - ik_z E_y)\underline{i} + \dots$

$$\nabla \wedge \underline{E} = i\underline{k} \wedge \underline{E}$$

2) a) $\underline{B}(r,t) = B_0 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$ $\quad$ HIL Medium.

Show Max's eqns reduce to:

$$\underline{k}\cdot\underline{E} = 0 \qquad \underline{k}\cdot\underline{B} = 0 \qquad \underline{B} = \frac{\hat{\underline{k}} \wedge \underline{E}}{v} \qquad E = -v(\hat{\underline{k}} \wedge \underline{B})$$

i) Sub in $\nabla \cdot \underline{E} = i\underline{k}\cdot\underline{E} = 0$ from Max's

ii) Sub in $\nabla \cdot \underline{B} = i\underline{k}\cdot\underline{B} = 0$ from Max.

iii) $\nabla \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ (max) so $i\underline{k} \wedge \underline{E} = i\omega \underline{B}$ $\quad$ not a vacuum..

$\hat{\underline{k}} = \omega/v$

$$B = \frac{1}{\omega}\underline{k} \wedge \underline{E} = \frac{k}{\omega}\hat{\underline{k}} \wedge \underline{E} = \frac{1}{v}\hat{\underline{k}} \wedge \underline{E}$$

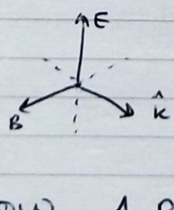
iv) $\nabla \wedge \underline{B} = \mu\mu\epsilon\epsilon \frac{\partial \underline{E}}{\partial t}$ (max) $\quad$ so $\quad i\underline{k} \wedge \underline{B} = -\mu\mu\epsilon\epsilon\, i\omega \underline{E}$

$$\underline{E} = -\frac{1}{\mu\mu\epsilon\epsilon\,\omega}\underline{k} \wedge \underline{B} = -\frac{1}{\epsilon\epsilon\mu\mu\,\omega} k\, \hat{\underline{k}} \wedge \underline{B}$$

$$= -v^2 \frac{1}{v} \hat{k} \wedge B = -v\hat{\underline{k}} \wedge \underline{B}$$

b) Show $E \perp B$.

$\underline{B} = \frac{1}{v}\hat{\underline{k}} \wedge \underline{E}$. $\quad$ Shows $\underline{k} \perp \underline{E}$ & $\underline{k} \perp \underline{B}$. $\quad \underline{k}\cdot\underline{E} = 0$ so perp.



c) Show $\wedge$ product relations obtained from Faraday's/Ampere's Law are equivalent.

from 2 iii) & iv) $\quad \underline{B} = \frac{1}{v}\hat{k} \wedge E$

$$\hat{\underline{k}} \wedge \underline{B} = \frac{1}{v}\hat{\underline{k}} \wedge (\hat{\underline{k}} \wedge \underline{E})$$

$= \frac{1}{v}(\underline{\hat{k}}(\underline{\hat{k}}\cdot\underline{E}) - \underline{E}(\hat{k}\cdot k))$

$\uparrow_0 \qquad \uparrow_1$

$\hat{\underline{k}} \wedge \underline{B} = -\frac{1}{v}\underline{E}$

hence $\underline{E} = -v\,\hat{\underline{k}} \wedge \underline{B}$ so Ampère's gay!! (with Maxwell)

3)a) $\underline{E}(z,t) = E_0\cos(kz-\omega t)\hat{x} - E_0\sin('')\hat{y}$ > ~~Eighth~~ and

use $\underline{B} = \sqrt{\mu_0\epsilon_0}\,(\hat{k}\times E)$ to find $\underline{B}$ field.

$\underline{B} = \sqrt{\mu_0\epsilon_0}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \hat{k}_x & \hat{k}_y & \hat{k}_z\\ E_x & E_y & E_z\end{vmatrix} = \dots = \sqrt{\mu_0\epsilon_0}\,E_0\,(\sin\hat{x} + \cos\hat{y})$

$\sin(kz-\omega t)\,\hat{x}$ btw

or $E_0\begin{vmatrix}\hat{x} & y & \hat{z}\\ 0 & 0 & 1\\ \cos & -1 & 0\end{vmatrix}$

$\underline{k} = (0,0,k) \quad \hat{\underline{k}} = (0,0,1)$

$\underline{B} = \frac{E_0}{c}(\sin(kz-\omega t), \cos(kz-\omega t), 0)$

now they're $\perp$.

b) magnitudes of $\underline{E}$ & $\underline{B}$, show they're constant independant of z & t.

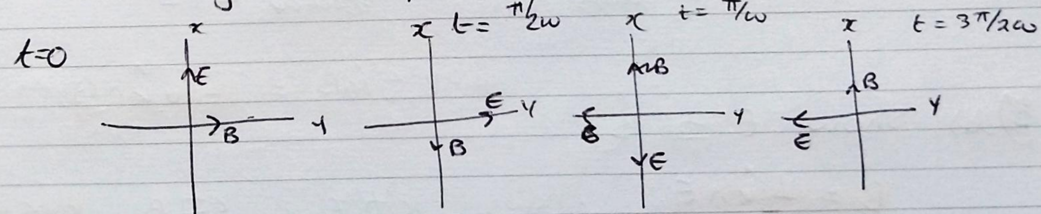
$|E|^2 = E_0^2(\underbrace{\cos^2 + \sin^2}_{=1} + 0^2) = E_0^2$

$|\underline{B}|^2 = \frac{E_0^2}{c^2}(\underbrace{\sin^2 + \cos^2}_{=1} + 0^2) = E_0^2/c^2$

no zs and ts!

$\sin(kz-\omega t)$. btw.

c) Draw diagrams of these at $t = 0, \pi/2\omega, \pi/\omega, 3\pi/2\omega$.



E_0 or $\frac{E_0}{c}$ $\cos(-\omega t), \sin(-\omega t)$. depending on field

d) modify for left hand polarisation.

$\underline{E} = E_0(\cos(kz-\omega t), \sin, 0)$

$\underline{B} = \frac{E_0}{c}(-\sin, \cos, 0)$

e) Calculate Poynting. $\underline{S} = \underline{E} \wedge \underline{H}$ = constant vector.

$= \frac{\underline{E} \wedge \underline{B}}{\mu_0}$

$$\underline{S} = \frac{E_0^2}{\mu_0 c} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos & -\sin & 0 \\ \sin & \cos & 0 \end{vmatrix} = \frac{E_0^2}{\mu_0 c} (\cos^2 + \sin^2) \hat{\underline{z}} = \boxed{\frac{E_0^2}{\mu_0 c} \hat{\underline{z}}}$$

constant

4) $\nabla^2 \underline{E} = \mu_r \mu_0 \sigma \frac{\partial \underline{E}}{\partial t}$

a) find k value. Sub $\underline{E}(z,t) = E_0 e^{i(kz - \omega t)} \hat{x}$ in.

$\nabla^2 E = -k^2 E = -\mu_r \mu_0 \sigma i \omega \underline{E}$ $\quad k = \sqrt{i}\sqrt{\mu_r \mu_0 \sigma \omega}$

$\sqrt{i} = \frac{1}{\sqrt{2}}(1+i) \quad = (1+i)\sqrt{\frac{\mu_r \mu_0 \sigma \omega}{2}} = (1+i) k_c$

$$\underline{E} = E_0 \underbrace{e^{-k_c z}}_{\text{real}} \underbrace{e^{i(k_c z - \omega t)}}_{\text{imaginary}}$$

$$\boxed{k_c = \sqrt{\frac{\mu_r \mu_0 \sigma \omega}{2}}}$$

b) what is λ_c in a conductor? what's $\lambda_c : \lambda$ in dielectric

$\lambda_c = 2\pi/k = 2\pi \sqrt{\frac{2}{\mu_r \mu_0 \sigma \omega}}$ $\quad \boxed{\text{from } \nabla^2 E = \mu_r \mu_0 \sigma \frac{\partial E}{\partial t}}$

$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi}{\omega \sqrt{\mu_r \mu_0 \varepsilon_r \varepsilon_0}}$

$\lambda_c/\lambda = \boxed{\sqrt{\frac{2 \varepsilon_r \varepsilon_0 \omega}{\sigma}}}$ small

5) $\mu_r \sim 1$, $\sigma \sim 6 \times 10^7$ thickness of walls = ? factor of 200

Amplitude reduces to $e^{-z/\delta}$ $\quad \delta = \sqrt{\frac{2}{\mu_r \mu_0 \sigma \omega}}$

$e^{-z/\delta} = 1/200 \quad z = \ln 200 = \sqrt{\frac{2}{\mu_0 \sigma_0 \omega}} \times 5.3 = \boxed{0.109 \text{ mm}}$

$\ln 200$

$f = 10 \times 10^6$
change to ω